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A Uniqueness Theorem for $y' = f(x, y)$ Using a Certain Factorization of f

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1. INTRODUCTION

Let R denote the reals, and let $x_0, x_1 \in R$, $x_0 < x_1$, $I = (x_0, x_1)$, and $S = \bar{I} \times R$. Consider the real-valued function $f: S \to R$ and the initial-value problem:

$$\begin{aligned} &\text{(i)} \qquad y'(x) = f(x, y(x)), \qquad x \in I; \\ &\text{(ii)} \qquad y(x_0) = y_0 . \end{aligned} \tag{P}$$

By a solution to (P), one will mean a function $y \in C(\bar{I})$ which is differentiable on I and satisfies (i), and satisfies (ii).

The concern here will be only with the question of uniqueness for (P) and a theorem will be proved which, as will be seen, assures uniqueness when many of the standard Lipschitz type uniqueness theorems do not apply. The "standard" theorems referred to here are those which explicitly impose conditions on the difference $f(x, y_2) - f(x, y_1)$ by way of generalizing the usual Lipschitz-continuity requirement for f. For examples of these theorems, reference is made to the uniqueness theorems of Diaz and Walter [1], Osgood [2], Tonelli [3], Montel [4], Kamke [5], and Brauer and Sternberg [6].

Consider the following two examples which will be used to demonstrate the applicability of the theorem below.

EXAMPLE 1. Consider the initial value problem

$$\begin{aligned} y'(x) &= g(x, y) = (1 - x)|\, y(x)|^{1/2}, \qquad 0 < x < 1 \\ y(0) &= 0, \end{aligned}$$

which has as two of its solutions

$$y(x) \equiv 0 \qquad \text{on} \qquad [0, 1],$$

and

$$y(x) = \begin{cases} 0, & 0 \leqslant x \leqslant \frac{1}{2} \\ \frac{1}{4}[x - \frac{1}{2}x^2 - \frac{3}{8}]^2, & \frac{1}{2} \leqslant x \leqslant 1. \end{cases}$$

EXAMPLE 2. Consider the initial value problem

$$\begin{aligned} y'(x) &= f(x, y) = (1 - x) \cdot (|y(x)|^{1/2} + 1), \qquad 0 < x < 1 \\ y(0) &= 0. \end{aligned}$$

It is clear that for any pair y_1, y_2 and any $x \in [0, 1]$,

$$f(x, y_2) - f(x, y_1) = g(x, y_2) - g(x, y_1). \tag{1}$$

Therefore, if one attempts to check the uniqueness in Example 2 by using any of the theorems in [1]–[6] one is bound to fail because their assertion of uniqueness for Example 2 would be an assertion of uniqueness for Example 1 by way of Eq. (1). Hence, the above theorems apparently do not apply to Example 2.

The solution in Example 2 is, however, unique and is a special application of the theorem which follows.

2. THE UNIQUENESS THEOREM

DEFINITION. A function $f: S \to R$ is said to satisfy a *one-sided, Lip*(x) condition on the rectangle $R_a = I \times [-a, +a]$ if $\exists$ a nonnegative function $k \in L_1\{I\} \ni$ for all $(x, y_1), (x, y_2) \in R_a$ with $y_2 > y_1$ it is true that

$$f(x, y_2) - f(x, y_1) \leqslant k(x)(y_2 - y_1).$$

A function f satisfying this condition will simply be denoted by writing $f \in \mathrm{Lip}(x)^+$ on R_a.

THEOREM. Suppose that it is possible to find three functions $f_1, f_2, f_3 : S \to R$ such that $f \equiv f_1 \cdot f_2 - f_3$ on S, where $f_1, f_3 \in C\{S\}$, and $\partial f_1(x, y)/\partial x$ exists, is nonnegative, and is dominated by a Lebesgue integrable function of y on each rectangle $I \times [-a, +a]$. Further suppose that f_1, f_2, f_3 have the properties:

(i) f_1 is strictly positive along solution curves;

(ii) for fixed $x \in I$,

$$-\infty < y_1 < y_2 < +\infty \Rightarrow f_3(x, y_2) f_1(x, y_1) \geqslant f_3(x, y_1) f_1(x, y_2);$$

(iii) $f_2 \in \mathrm{Lip}(x)^+$ on each rectangle $I \times [-a, +a]$.

Then there exists at most one solution to (P).

Remark 1. This theorem applies to the above example by taking

$$f_1 \equiv 1 + |y|^{1/2}, \qquad f_2 \equiv 1, \qquad f_3 \equiv x(1 + |y|^{1/2}).$$

Remark 2. None of (i) through (iii) may be dropped. If one considers the popular example of nonuniqueness

$$\begin{aligned} y'(x) &= f(x, y) = |y|^{1/2}, \qquad 0 < x < 1 \\ y(0) &= 0 \end{aligned}$$

then one sees the following. By choosing:

(i′) $\quad f_1 \equiv |y|^{1/2}, \quad f_2 \equiv 1, \quad f_3 \equiv 0;$

(ii′) $\quad f_1 \equiv 1 + |y|^{1/2}, \quad f_2 \equiv 1, \quad f_3 \equiv 1;$

(iii′) $\quad f_1 \equiv 1, \quad f_2 \equiv |y|^{1/2}, \quad f_3 \equiv 0;$

in each case all the hypotheses of the theorem are satisfied except respectively (i), (ii), and (iii), while the result in each case is nonuniqueness.

Remark 3. Clearly if the function f itself satisfies the $\mathrm{Lip}(x)^+$ condition then the remainder of the hypotheses are trivially satisfied and we obtain a corollary which the additional assumption that f be continuous, appears in Walter [7]. Obviously, a corollary to this is the usual Lipschitz theorem.

Remark 4. If the function f satisfies the requirements imposed on f_1 in the theorem, we obtain a corollary which bears a similarity to a result of Wend [8] which states that if (i) $x_2 > x_1$ implies $f(x_2, y) \geqslant f(x_1, y)$, (ii) f is nonnegative, (iii) $f > 0$ on solution curves, then (I) has at most one solution. Notice that Wend's result does not apply to Example 2 since (i) cannot be satisfied.

Proof of the Theorem. Suppose, contrary to what is to be shown, that there are two solutions u and v to (P) and there exists a point

$$\bar{x} \in I \ni u(\bar{x}) - v(\bar{x}) > 0.$$

Let

$$\xi_0 = \sup\{x : x_0 \leqslant x < \bar{x} \qquad \text{and} \qquad u(x) = v(x)\}.$$

Then $u(\xi_0) = v(\xi_0)$ and $u(x) > v(x)$ on $(\xi_0, \bar{x})$. Now $f_1[\xi_0, v(\xi_0)] > 0$ and since f_1 is continuous there exists a ξ_1 such that for $\xi_0 \leqslant x \leqslant \xi_1$ and $v(x) \leqslant y \leqslant u(x)$ it is true that $f_1(x, y)$ is bounded away from zero by a positive number m.

Define a function $H : [\xi_0, \xi_1] \to R$ by

$$H(x) = \int_{v(x)}^{u(x)} \frac{dy}{f_1(x, y)} + \int_{\xi_0}^{x} \left\{ \frac{f_3[y, u(y)]}{f_1[y, u(y)]} - \frac{f_3[y, v(y)]}{f_1[y, v(y)]} \right\} dy. \tag{2}$$

Using the positivity of f_1 together with (ii), it follows that

$$H(x) > 0 \quad \text{on} \quad (\xi_0, \xi_1]. \tag{3}$$

Also, by choosing large enough a, there exists $g \in L_1\{[-a, +a]\} \ni$

$$0 \leqslant [\partial f_1(x, y)/\partial x] \leqslant g(y) \; \forall x \in I, \qquad v(x) \leqslant y \leqslant u(x). \tag{4}$$

Because of (4), $H(x)$ may be differentiated on (ξ_0, ξ_1) to obtain

$$\begin{aligned} H'(x) &= \int_{v(x)}^{u(x)} \frac{\partial}{\partial x} \left[\frac{1}{f_1(x, y)} \right] dy + \frac{u'(x)}{f_1[x, u(x)]} - \frac{v'(x)}{f_1[x, v(x)]} \\ &\quad + \frac{f_3[x, u(x)]}{f_1[x, u(x)]} - \frac{f_3[x, v(x)]}{f_1[x, v(x)]} \\ &\leqslant \frac{u'(x) + f_3[x, u(x)]}{f_1[x, u(x)]} - \frac{v'(x) + f_3[x, v(x)]}{f_1[x, v(x)]}. \end{aligned} \tag{5}$$

Using the factorization of f together with the Lip$(x)^+$ condition, it follows that

$$H'(x) \leqslant f_2[x, u(x)] - f_2[x, v(x)] \leqslant k(x) \cdot [u(x) - v(x)]. \tag{6}$$

Now letting M be the maximum of f_1 on $\xi_0 \leqslant x \leqslant \xi_1$, $v(x) \leqslant y \leqslant u(x)$, one has

$$H(x) \geqslant \int_{v(x)}^{u(x)} \frac{dy}{M} = \frac{u(x) - v(x)}{M}.$$

Combining this with (6), one finally obtains the linear differential inequality

$$H'(x) - Mk(x)\, H(x) \leqslant 0 \; \forall x \in (\xi_0, \xi_1),$$

which, upon introducing the usual integrating factor, may be written as

$$\frac{d}{dx} \left[H(x) \exp - \int_{\xi_0}^{x} Mk(t)\, dt \right] \leqslant 0 \qquad \text{a.e. on} \quad (\xi_0, \xi_1).$$

Now, denoting by D^+ the upper right-hand derivate, it is obvious that

$$D^+ \left[-H(x) \exp - \int_{\xi_0}^{x} Mk(t)\, dt \right] \geqslant 0 \qquad \text{a.e. on} \quad (\xi_0, \xi_1), \tag{7}$$

and it follows with little difficulty that

$$D^+ \left[-H(x) \exp - \int_{\xi_0}^{x} Mk(t)\, dt \right] > -\infty \qquad \forall x \in (\xi_0, \xi_1). \tag{8}$$

Applying a weak form of the Fundamental Theorem, (7) and (8) imply that

$$\begin{aligned} 0 &\leqslant \int_{\xi_0}^{x} \frac{d}{dt} \left[-H(t) \exp - \int_{\xi_0}^{t} Mk(s)\, ds \right] dt \\ &\leqslant -H(x) \exp - \int_{\xi_0}^{x} Mk(s)\, ds + H(\xi_0). \end{aligned}$$

Since $H(\xi_0) = 0$, this implies that $H(x) \leqslant 0$ on (ξ_0, ξ_1), contradicting (3)$\|$.

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